

Expected Properties of Massive Neutrinos for Mass Matrices with a Dominant Block and Random Coefficients Order Unity

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Abstract

We study the class of neutrino mass matrices with a dominant block but unspecified $\mathcal{O}(1)$ coefficients, and scan the possible models by the help of random number generators. We discuss which are the most common expectations in dependence of the adjustable parameter of the mass matrices, ε , and emphasise an interesting sub-class of models that have large mixing angles for atmospheric and solar neutrinos, and an angle θ_{13} close to the experimental limit. For those models where the lepton mass matrices are subject to Froggatt-Nielsen U(1) selection rules, we show that the neutrino mixing matrix receives important contributions from the rotations operating on charged lepton sector, which increase the predicted value of the angle θ_{13} and the ee-entry of the neutrino mass.

A specific, simple form of neutrino mass matrix has stimulated many studies in the last few years (see *e.g.* [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]):

$$\mathbf{M}_\nu \propto \text{diag}[\varepsilon, 1, 1] \mathcal{O}(1) \text{ diag}[\varepsilon, 1, 1] \quad (1)$$

where it is understood that ε is a small parameter¹, and $\mathcal{O}(1)$ is the matrix of the “coefficients of order unity” (an argument for eq. (1) with a minimum of theoretical assumptions is presented in [12]). The attempt to understand

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¹The case $\varepsilon = 1$ has been given theoretical support in [11], and was discussed in detail in [6] and [7]; however, we regard it as an extreme case.

the gross features of the fermion (neutrino) mass matrices by the help of factor ε but without stressing too much the role of the coefficients is in the line of thought drawn by Froggatt and Nielsen [13]. In their language, the matrix in eq. (1) is characterised by the presence of two degenerate U(1) charges, and this implies that a block of the mass matrix has relatively large entries (“dominant block”). Recently, the early suggestion [13] that the coefficients should be thought as random numbers was taken quite literally in the studies [6, 7, 9, 10]. We find this type of approach of interest, since it amounts to a scan of the possible theoretical models; the most frequent cases (in dependence on the value of ε) are emphasised in this approach. We show in the appendix our assumptions on the coefficients, seen as random numbers, and discuss related problems.

We begin by discussing the value of ε putting emphasis on available neutrino oscillation data. Postponing the interpretation of LSND indications, we know with good confidence that² the atmospheric mixing angle θ_{23} is in the range $45^\circ \pm 10^\circ$, while θ_{13} belongs roughly to $(0 - 10)^\circ$; the corresponding Δm_{31}^2 is in the range $(1.5 - 5) \times 10^{-3}$ eV². (It is rather evident that the two informations on the mixing angles are in reasonable agreement with the mass matrix in eq. (1), if ε is sufficiently small). The situation with solar neutrino observations is less clear. We assume that the observations indicate again oscillations of massive neutrinos, with $\Delta m_{21}^2 \ll \Delta m_{31}^2$, and select three regions of parameter space for further discussion: (1) a SMA region, namely the rectangle with vertices $(4 \times 10^{-6}, 2 \times 10^{-4})$ and $(1 \times 10^{-5}, 3 \times 10^{-3})$ in the $(\Delta m_{21}^2/\text{eV}^2, \tan^2 \theta_{12})$ plane, (2) a LMA region, namely the rectangle with vertices $(8 \times 10^{-6}, 0.15)$ and $(3 \times 10^{-4}, 0.75)$ in the same plane, (3) a LOW (up to quasi-vo) region, that is the rectangle with vertices $(6 \times 10^{-10}, 0.3)$ and $(3 \times 10^{-7}, 3)$. Indeed, since these models do not predict the overall mass scale, we use Δm_{21}^2 to calculate values of the *hierarchy factor*:

$$h = \frac{\Delta m_{21}^2}{\Delta m_{31}^2}$$

assuming that $\Delta m_{31}^2 = 3 \times 10^{-3}$ eV², and then we compare these values of h with the calculated ones. The percentage of mass matrices that passes the cuts on $\theta_{23}, \theta_{13}, \theta_{12}$ and h is shown in fig. 1. As pointed out in [6, 7],

²As usual, $\mathbf{M}_\nu = U^* \text{diag}[m_j \times \exp(i\xi_j)] U^\dagger$, $m_j \leq m_{j+1}$ and $\Delta m_{ji}^2 = m_j^2 - m_i^2$. In terms of the neutrino mixing matrix U (such that $\nu_\ell = U_{\ell i} \nu_i$, with $\ell = e, \mu, \tau$ and $i = 1, 2, 3$) the mixing angles θ_{ij} , with values $0^\circ \leq \theta_{ij} \leq 90^\circ$, are simply $\sin \theta_{13} = |U_{e3}|$, $\tan \theta_{12} = |U_{e2}/U_{e1}|$ and $\tan \theta_{23} = |U_{\mu 3}/U_{\tau 3}|$.

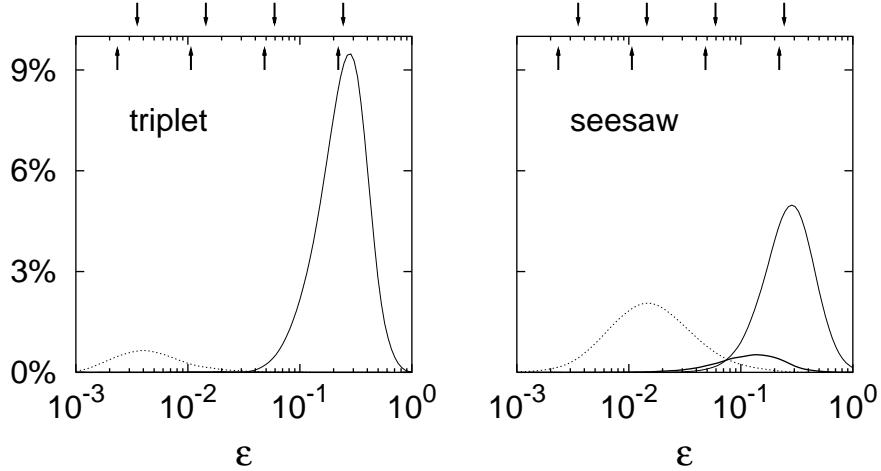


Figure 1: Percentage of phenomenologically successful neutrino mass matrices of the type eq. (1) as a function of ε . Dashed line denotes the SMA region, continuous thin line the LMA region, thick line the LOW region (last one is practically invisible for triplet case). For orientation, we emphasise with arrows pointing downward the special cases $\varepsilon = (m_\mu/m_\tau)^{0.5,1,1.5,2}$ and with those pointing upward the cases $\varepsilon = (\sin \vartheta_C)^{1,2,3,4}$ (ϑ_C = Cabibbo angle).

it is of some importance to distinguish the case denoted as “triplet” in the figure, when one generates a matrix with random coefficients in eq. (1) (namely $\mathcal{O}(1) = R_1$; see [14] for theoretical support) from the case when $\mathcal{O}(1) = R_2 R_3^{-1} R_2^t$, which corresponds to assume that the light neutrino masses are due to the “seesaw” mechanism [15]. In last case, the random coefficients are those of the Dirac neutrino couplings, and of the heavy (right handed) neutrinos R_2 and R_3 respectively.³ It is seen that a reasonably large number of mass matrices of the type eq. (1) survive the cuts for certain values of ε . It is rather obvious that for very large values of ε the number of successful models decreases due to the cut on θ_{13} . The lower peak height of the SMA- (even more, LOW-) curves is due to the difficulty to reduce sufficiently h (which is alleviated in the “seesaw” case [6, 7]). In both cases, there is a gap with relatively small overlap between the SMA and LMA curves. This can be explained in the following manner: consider the plane (h, θ_{12}) ;

³In general, both contributions are present: $\mathcal{O}(1) = \sin^2 \eta R_1 + \cos^2 \eta R_2 R_3^{-1} R_2^t$. For simplicity, we will discuss only the extreme cases—triplet and seesaw—but it is clear that we can bias this or that value of η only adding theoretical information.

Q_e	Q_μ	Q_τ	Q_{e^c}	Q_{μ^c}	Q_{τ^c}	ε (degrees)
3	0	0	3	2	0	.83°
2	0	0	4	2	0	3.4°
1	0	0	5	2	0	14.°

Table 1: Three sets of Froggatt-Nielsen leptonic charges which give an ascending series of $\varepsilon = (v/M)^{Q_e}$ values (v/M is discussed in the text). We normalise to 0 the lowest charges since we focus only on charged lepton mass ratios; $Q_{\mu^c} = 2$ is needed to reproduce m_μ/m_τ ; and $Q_e + Q_{e^c} = 6$ (or nearby values) to reproduce $m_\mu/m_\tau = (m_e/m_\tau)^{1/3}$ (correct at the 10 % level).

as typical values, we have $h \sim 1$ and $\theta_{12} \sim \varepsilon$, but there is an interesting tail (=a less populated region) which shrinks with ε where θ_{12} increases and h decreases (the correlation being tighter in the “triplet” case). For diminishing values of ε , the tail first meets the LMA region and only after moves toward the SMA region: this creates the gap observed in fig. 1.⁴ The different position and height of the “SMA-peak” from seesaw to triplet case indicate just that the distributions (and tails) are different.

Some models of this type (=values of ε) have been discussed already in the literature: $\varepsilon = \sin^3 \vartheta_C$ in [1], $= m_\mu/m_\tau$ in [2, 7], $= \sin^n \vartheta_C$ in [4], 1 in [6]. Except than in the last case, the stress was put on the correlations of the properties of neutrinos with those of charged fermions. However, no special emphasis has been put on the case:

$$\varepsilon = \sin \vartheta_C \approx 0.22 \quad \text{or} \quad = (m_\mu/m_\tau)^{1/2} \approx 0.24, \quad (2)$$

that we see to be quite interesting in connection with the LMA region (in both the “seesaw” and the “triplet” case). The case $\varepsilon = m_\mu/m_\tau \approx 0.06$ was emphasised in reference [7]; it was shown that the use of U(1) selection rules *a la* Froggatt and Nielsen, with charges $Q_e = 1$, $Q_\mu = Q_\tau = 0$ for the left fields is not in contradiction with the gross features of the charged lepton spectrum, if at the same time one assumes $Q_{e^c} = 2$, $Q_{\mu^c} = 1$ and $Q_{\tau^c} = 0$ as charges for the right fields. The same can be however obtained with other

⁴This is the same argument put forward in sect. 2 of ref. [4]. But note that eq. (2.1) therein is improved assuming that the 33-entry is 2, not 1; at the same time, $m_\nu \sim (\Delta m_{atm}^2)^{1/2}$ has to be replaced by $m_\nu \sim (\Delta m_{atm}^2)^{1/2}/2$. This affects the numerical factor in front to eqs. (2.2-4), and the region in figure 1 shifts downward by a factor of 4.

choices of charge, as those in table 1 with a different value of the parameter that regulates all mixings and hierarchies, $v/M = (m_\mu/m_\tau)^{0.5}$ (note that the second model is just the one of Sato and Yanagida [7], since $Q \rightarrow 2Q$ but also $v/M \rightarrow (v/M)^{1/2}$ in our table).

We verified numerically the viability of the alternative choices, and found that increasing Q_{e^c} by 1 unity also leads to similar results, for slightly larger values of ε ; similar conclusions even decreasing Q_{e^c} by 1 unity (but worsening a bit the agreement with charged fermion mass ratios in the first two cases of table 1): Nothing against large values of ε as in eq. (2) from this side. Indeed, it was argued already in [13] that the value of v/M should be large, simply in connection with the size of the Cabibbo angle; $\sin \vartheta_C$ should presumably come out as the result of a fluctuation, if v/M were much smaller.

Now we tackle one last problem, which is strictly linked to the theoretical (Froggatt-Nielsen) context. We start with some general consideration: One can attempt to classify the models for neutrino masses, depending on whether the neutrino mixing matrix

$$U = U_E^\dagger U_\nu \quad (3)$$

is 1. mostly due to the rotation of the neutral leptons U_ν , or 2. to the rotation of charged leptons U_E , or, finally, 3. if both have a comparable importance (at least for some mixing angle). It is easy to argue for the first possibility (at least in words); neutrinos look special, charged leptons could always entail small mixing, and thence U_ν could play an overwhelming role in U . As an example of the second possibility, we quote the “lopsided” models described in [16]. Finally, as an important instance of the last possibility, U_E and U_ν are typically of comparable importance for U in the U(1) approach of Froggatt and Nielsen [13], since they are controlled by the same charges, namely, those of the left leptonic fields (note, however, that a partial degeneracy in the neutrino mass matrix might enhance the role of U_ν). Strictly speaking, most of the considerations above apply to the first case, in that (following previous studies) we simply ignored the role of the mixing due to the charged leptons ($U_E \approx \mathbb{1}$). However, since the approach with random numbers is naturally (though not unavoidably) connected with the use of U(1) selection rules for fermion mass matrices, and since the existing studies of neutrino mass matrices of this type [7, 6, 9, 10] do not take into account the point, we decided to investigate what is the effect of U_E in models with U(1) selection rules. Thence, we additionally generated the random mass matrices R_0 as

$.83^\circ$	$45 - \theta_{23}$	θ_{13}	θ_{12}	h	$m_{ee}/10^{-4}$	SMA	LMA	LOW
t,w/o	± 12	$.37 \pm .19$	1.0 ± 1.4	$.35 \pm .26$	1.4 ± 3.3	.15	.00	.01
t,w	± 23	$.70 \pm .33$	1.2 ± 1.4	$.35 \pm .26$	2.9 ± 1.7	.04	.00	.00
s,w/o	± 17	$.52 \pm .29$	1.3 ± 1.7	$.12 \pm .16$	1.4 ± 1.3	2.0	.00	.02
s,w	± 21	$.79 \pm .41$	1.5 ± 1.7	$.12 \pm .16$	2.9 ± 2.4	.92	.00	.01
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3.4°					$m_{ee}/10^{-3}$			
t,w/o	± 12	1.5 ± 0.8	3.8 ± 3.8	$.35 \pm .26$	2.4 ± 0.6	.01	.48	.01
t,w	± 23	2.9 ± 1.4	4.6 ± 3.8	$.35 \pm .26$	4.9 ± 2.9	.00	.12	.00
s,w/o	± 17	2.1 ± 1.2	5.0 ± 5.0	$.12 \pm .16$	2.3 ± 2.1	.58	.18	.27
s,w	± 21	3.3 ± 1.7	5.7 ± 5.1	$.12 \pm .16$	4.9 ± 4.0	.24	.15	.07
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$14.^\circ$					$m_{ee}/10^{-2}$			
t,w/o	± 12	6.2 ± 3.2	12.5 ± 8.4	$.36 \pm .26$	4.0 ± 0.9	.00	9.1	.00
t,w	± 23	11.8 ± 5.6	16.3 ± 9.3	$.36 \pm .26$	7.9 ± 4.6	.00	9.9	.00
s,w/o	± 17	8.7 ± 4.6	17.1 ± 12.3	$.13 \pm .17$	3.7 ± 3.1	.03	4.8	.32
s,w	± 21	13.1 ± 6.6	20.0 ± 12.6	$.13 \pm .17$	7.6 ± 5.9	.01	2.0	.04

Table 2: Calculated neutrino properties for the U(1) models of tab. 1, in the cases with triplet or seesaw (t and s resp.) mass mechanism, and with or without the account of the lepton mixing matrix U_E (w and w/o resp.). The 3 parts of the table correspond to the models defined in tab. 1 (in the left-upper corners, the values of ε in degrees). All angles in the table are in degrees, while SMA, LMA and LOW denote percentages (cuts as in fig. 1).

coefficients of the charged lepton mass matrix $\mathbf{M}_E = U_E^* \text{ diag}[m_\ell] V_E$:

$$\mathbf{M}_E \propto \text{diag} \left[\left(\frac{v}{M} \right)^{Q_{\ell'}} \right] \mathcal{O}(1) \text{ diag} \left[\left(\frac{v}{M} \right)^{Q_{\ell c}} \right]$$

(where $\ell, \ell' = e, \mu, \tau$) enforcing the values of the U(1) charges in tab. 1 to specify the models fully. Only if the mass ratios (m_e/m_τ) and (m_μ/m_τ) are reproduced within 30 %, we calculate U_E and estimate its effect on U in eq. (3) (it is important to implement such a condition to gauge out cases when the mixing angles come out artificially large).

The results are presented in table 2. The spread⁵ in θ_{23} does not change much with ε , however, it increases significantly with the inclusion of U_E

⁵The spread δx on the quantity x is calculated as $\delta x^2 = \sum_i^N (x_i - \langle x \rangle)^2 / N$, where x_i is the result of the i^{th} simulation, $\langle x \rangle = \sum_i^N x_i / N$ is the average, and $N = 10^6$.

effects; it is a pity that these simple models are unable to give an indication on the size of the deviation of θ_{23} from 45° , one of the most interesting quantities to be searched in future experiments. The average values of θ_{12} and θ_{13} also increase with the inclusion of U_E . If U_E effects are not included, θ_{13} is on average smaller than ε [7]; the reason is just the numerical factor ~ 2 discussed in footnote 4. The angle θ_{13} is rather close to ε when these effects are included, and close to the experimental limit when ε is large. Note that even a seemingly modest increase in θ_{13} , say by a factor of 2 (table 2, triplet case) is an important message for the searches of ν_e appearance in terrestrial experiments, since the probabilities of conversion in vacuum depend strongly on this parameter: $P(\nu_\mu \rightarrow \nu_e) \propto (\theta_{13})^2$. In tab. 2 we also present the value of the following quantity:

$$m_{ee} = \frac{|(\mathbf{M}_\nu)_{ee}|}{(\Delta m_{31}^2)^{1/2}}$$

These calculations show that, for the larger value of ε considered, the ee-element of the mass matrix $|(\mathbf{M}_\nu)_{ee}|$ can reach the several meV level. This is tantalisingly close to the expected sensitivity of the next generation neutrinoless double beta decay experiments, namely $10 - 20$ meV.

Let us summarise and discuss the results. Neutrino masses are expected to be non-zero, and indeed the atmospheric neutrino mass scale $\sim 50 - 60$ meV is very well compatible with the ideas of grand unification, but the size of the mixing angles is a puzzle, in particular the strong indication that at least one of them is large—maybe maximal. We discussed a class of neutrino mass models (eq. (1)) that are inspired to the principle that all elements of the mass matrices (seen as fundamental quantities) should be equally large, unless explicitly suppressed. We emphasised the type of models that are phenomenologically successful, and discussed the stability in the choice of coefficients by the help of random number generators (fig. 1). We discussed also the connection with known properties of the charged fermions (and, in particular, leptons), and remarked that large values of the “order parameter” ε (see eq. (2)) are quite interesting in connection with the LMA solution of the solar neutrino problem. In present context, the triplet mechanism is not disfavored in comparison with the seesaw mechanism for neutrino mass generation; one could argue instead that the triplet mechanism is more predictive, for the simple reason that it favors solar neutrino solutions with as large h as possible. In close observance to the Froggatt-Nielsen approach, we also pointed out the relevance of the mixing due to charged leptons for

the neutrino mixing; this effect unfortunately renders almost random the expected value of the parameter θ_{23} in these models, though the maximal mixing $\theta_{23} = 45^\circ$ is the most likely possibility. Large values of the mixing angle θ_{13} , and perhaps an observable $|(\mathbf{M}_\nu)_{ee}|$ can be naturally incorporated in the model, if the parameter ε is large as in eq. (2).

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Appendix: What is a Random Coefficient?

In the spirit of the approach, the random coefficients of the mass matrices should not be enumerated among the parameters; however, their form has to be fixed in order to perform the scan of possible models. To give an idea of the effect of changing the generator, we compare the percentage of successful models in (SMA,LMA,LOW)-regions for the model of Sato and Yanagida (seesaw case, U_E included—tab. 2, middle part, last line). Variations amounts to a factor of a few in most extreme cases: (0.24, 0.16, 0.11) % for Z_0 , (0.42, 0.22, 0.11) % for Z_1 . We denote with Z_δ the numbers in the complex plane with random phase, and with modulus uniformly distributed in an interval $[-\delta, \delta]$ around 1 (Z_1 is a circle in the complex plane, Z_0 the circumference). Following [7], we adopt in our study the choice Z_δ , with $\delta = 0.2$, clearly consistent with the view that the coefficients are “order unity”.

There is another point of ambiguity: How to treat a symmetric random mass matrix R (indeed, the matrices R_1 and R_3 described in the main text are symmetric). One possibility is to generate just the elements R_{ij} with $i \geq j$, and then set $R_{ji} = R_{ij}$; another one is to generate the full matrix, and then replace R_{ij} and R_{ji} by their average.⁶ Following the previous studies, we adopted the first prescription, and noted that the second prescription

⁶These are clearly different prescriptions; for instance, the sum of 2 numbers uniformly distributed is not uniformly distributed.

leads to results that differs little from those in tab. 2 (in the same case of previous paragraph); θ_{23} is the same; $\theta_{13} = (3.2 \pm 1.6)^\circ$, $\theta_{12} = (5.6 \pm 5.0)^\circ$, $h = .11 \pm .15$, $m_{ee} = (4.7 \pm 3.9) \times 10^{-3}$; finally, the SMA,LMA and LOW success percentages are 0.26, 0.10 and 0.08 %.

Now, a delicate aspect; in this work (following previous studies) we assumed that all coefficients $\mathcal{O}(1)$ are distributed in the same manner, but *a priori* it is unclear whether this assumption is fair. Indeed, if terms higher order in v/M arise through the exchange of virtual particles as suggested in [13], there might be not only a piling up of (v/M) 's but also of the coefficients themselves.⁷ We do not want to deny the interest of this simplifying assumption (that we adopt in this study), but we point out a risk of minor reliability of the predictions which depend essentially on higher order terms. From this point of view, the prediction of a relatively large value of h and of θ_{23} are more reliable than those on the other mixing angles; the one on $|(\mathbf{M}_\nu)_{ee}|$ has the greatest dependence on this crucial assumption, and might be considered, then, even less reliable.

A final warning, of more general nature: Mass matrices with random coefficients can help to emphasise certain possibilities that are compatible with present (lack of) information, but one should be careful to interpret the results of this type of calculations in “probabilistic” term. Indeed, if, for consistence, the absolute scale of the neutrino mass matrix was also let fluctuate; or also, if one required that the (masses of the) charged leptons were reproduced within experimental errors; *etc.*; the probability of success of these attempts would have been practically zero. Similarly, if a theory of the coefficients order unity were given, any “statistical” consideration (like the present ones) would have been much less relevant.

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